Computer Algebra Systems: Problems on the Edge

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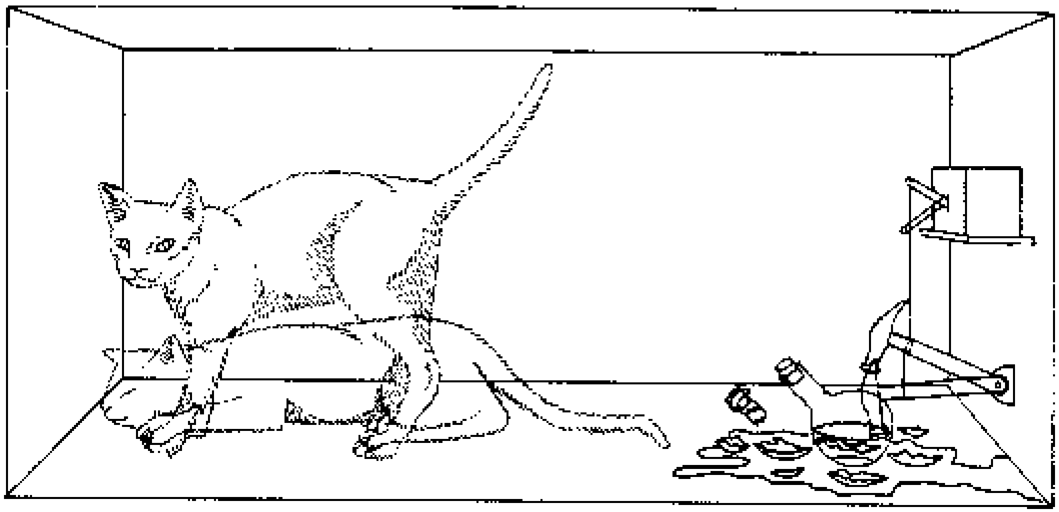
Schrödinger's Cat



Either the cat is alive or dead, but until we look into the box it is 50% alive and 50% dead

Solutions to a quadratic equation have a similar problem

- Solve (x^2-(a-b)x =0 for x);

$$\{\{x \to 0\}, \{x \to -a + b\}\}$$

- Solve(x^2-(a-b)x+c=0 for x);

$$\left\{\left\{x \to \frac{1}{2}\left(a + b - \sqrt{(a-b)^2 - 4c}\right)\right\}, \left\{x \to \frac{1}{2}\left(a + b + \sqrt{(a-b)^2 - 4c}\right)\right\}\right\}$$

If c=0, then the solutions are x→0 and x→(a-b).

- Solve(x^2-(a-b)x+c=0 for x), knowing that c=0.

$$x \to \frac{1}{2}\left(a - \sqrt{(a-b)^2} + b\right) \qquad x \to \frac{1}{2}\left(a + \sqrt{(a-b)^2} + b\right)$$

Is this x → 0 ?   Or is this x → 0 ?

You could choose either to be 0, in which case the other is a-b.

- Either solution in isolation is ambiguous. The pair is unique.

$$x \to \frac{1}{2}\left(a - \sqrt{(a-b)^2} + b\right) \qquad x \to \frac{1}{2}\left(a + \sqrt{(a-b)^2} + b\right)$$

Is this x → 0 ? Or is this x → 0 ?

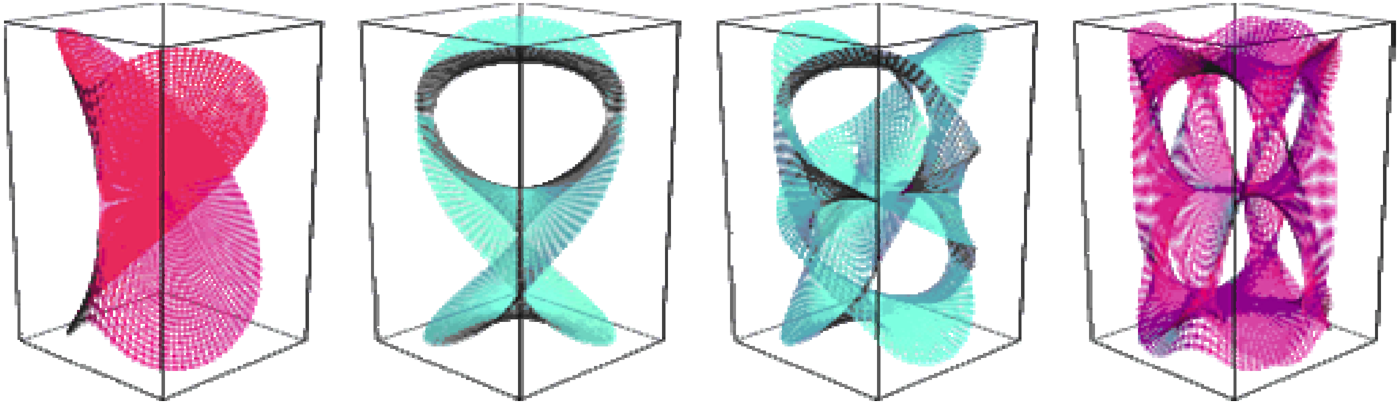
CAS often make a choice (or are forced to do so…)

Can we do this right?
Choose the “largest” solution. Is (a-b)>0 ??
Carry the solution as a set with +/- square root (in which case we can do some simplification).

$$x \to \frac{1}{2}\left(a - \sqrt{(a-b)^2} + b\right) \qquad x \to \frac{1}{2}\left(a + \sqrt{(a-b)^2} + b\right)$$

Is this x → 0 ? Or is this x → 0 ?

Can CAS do it right?



Is square root the only problem?
Clearly NOT. Branch cuts, Riemann surfaces make life tough.

Can programmers figure this out? Can mathematicians?
Mathematicians need to offer guidance
Programmers need more persuasive representations.
(see-- solutions of $w(z)^d+w(z)+z^{d-1}=0$, $d=2, 3, 4, 5$, w= Lambert's w function)

Have CAS fallen into a pit from which there is no escape?

- Sometimes the market, or a particular vendor sets the tone.
  - A solution is “good enough” in *Mathematica* *if it is wrong only on a domain of lower dimension than the solution.*
  - “Generic” solutions are offered e.g. $\int x^n dx$ is $x^{n+1}/(n+1)$ …ignoring the possibility that n=-1
  - Mma arithmetic is, to put it mildly, “unique”.

Short term alternatives

- Use unsophisticated libraries (no “knowledge” just algorithms on well-known domains like polynomials). Many such libraries, most open-source, are available now.
- Use computers very carefully.
  – Possibly with well-designed intelligent internet agents
  – Make sure you are not pushing beyond the already-solved
  – Seek help from users with comparable computations (internet search can help)

Longer term

- Participate in standards development for notation and semantics.
- Demand better software, perhaps developed as open source. Support intellectually and financially, new or continuing quality efforts.

Blame for Mathematicians, too

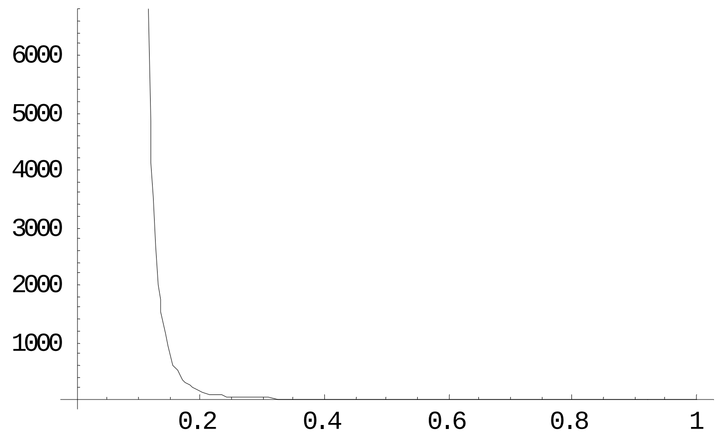
- Refusing to credit computers as credible partners in discovery.
- Failing to be algorithmic, even when that is plausible. Hiding exploratory efforts.
- Grasping at little straws. e.g. instead of writing an algorithm to do big X (e.g. "solve ODEs") discover a patch on a theorem about a particular, generally obscure abstraction. (little X)

Conclusions

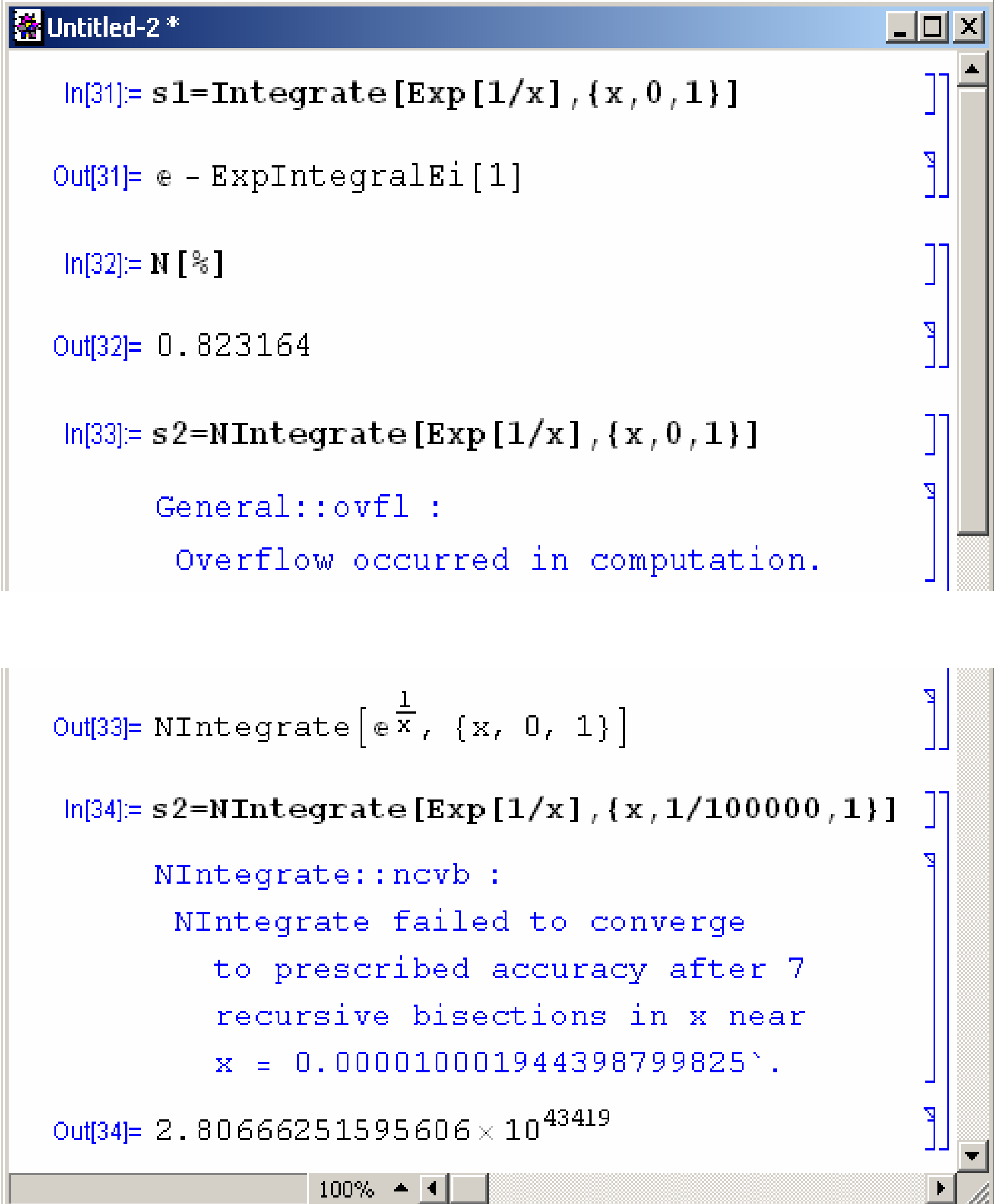
- Progress won't happen unless the next generation of mathematicians (and users of math: engineers, etc.) are more aware of the prospects of computing.
- Mathematicians (*present company excepted*) tend to be hostile to math mechanization ideas. Fix this!

Bug or Feature

- Even though we poke fun at Mma, there are often (the same or similar) bugs in other systems stemming from conceptual problem.
- This is just the most recent few we have encountered.



Exp(1/x)
0.1 to 1



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```
In[1]:= r1=Integrate[Exp[1/x],{x,10^(-6),1}]

Out[1]= e - e^1000000/1000000 - ExpIntegralEi[1] +
        ExpIntegralEi[1000000]

In[2]:= r2=Integrate[Exp[1/x],{x,10^(-8),1}]

Out[2]= e - e^100000000/100000000 - ExpIntegralEi[1] +
        ExpIntegralEi[100000000]

In[3]:= N[%1]

Out[3]= 3.033 × 10^434282

In[4]:= N[%2]

Out[4]= 0.

In[5]:= N[%1]-N[%2]

Out[5]= 3.033 × 10^434282

In[6]:= N[%1-%2]

Out[6]= 0.
```

Another example

Untitled-2 *

In[36]:= `1/(a+Sqrt[Cos[x]])^2`

Out[36]= $\dfrac{1}{\left(a+\sqrt{\text{Cos}[x]}\right)^2}$

In[37]:= `Integrate[%,{x,0,Pi/2}]`

Out[37]=
$$\frac{2}{-1+a^4} - \frac{2\,(1+a^4)\,\text{ArcTanh}\left[\frac{1+a^2}{\sqrt{1-a^4}}\right]}{\sqrt{1-a^4}\,(-1+a^4)} +$$
$$\Big(a\,(1-a^2)\,\Big(2\,a^2\,\text{EllipticK}[-1] -$$
$$2\,(-\text{EllipticE}[-1]+\text{EllipticK}[-1]) -$$
$$\Big((-1)^{1/4}\,\Big((1+a)\,\text{EllipticK}[2] - (1-i)$$
$$\text{EllipticPi}\left[\frac{(1-i)\,(-i-a)}{-1-a},\,2\right]\Big)\Big)\Big/$$
$$\Big(\sqrt{2}\,(-1-a)\,a\,(i+a)\Big) + 2\,a^4$$
$$\Big(\Big((-1)^{1/4}\,\Big((1+a)\,\text{EllipticK}[2] - (1-i)$$
$$\text{EllipticPi}\left[\frac{(1-i)\,(-i-a)}{-1-a},\,2\right]\Big)\Big)\Big/$$
$$\Big(\sqrt{2}\,(-1-a)\,a\,(i+a)\Big) -$$
$$\Big((-1)^{1/4}\,\Big((1-a)\,\text{EllipticK}[2] - (1-i)$$

100%

This answer is correct some places

- Correct for a=2
- Wrong for a=1/2, a=1.
- What tools do I have to examine the domain of validity using Mathematica?
- And why should I believe Mathematica once I see it is wrong!
- I could try Maple, or Macsyma; sometimes they make the same mistakes!

(Examples of other problems)

What is a summation?

$$\sum_{i=0}^{a} f(i) + \sum_{i=a+1}^{n} f(i) = \sum_{i=0}^{n} f(i)$$

What if you want this to be true even if 0<a or a>n ? Too bad if you are using Mathematica.